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L^2 Approximation with Trigonometric n-nomials

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We compare the degree of approximation to $L^2(-\pi, \pi)$ by nth degree trigonometric polynomials, with the degree of approximation by trigonometric n-nomials, which are linear combinations, with constant (complex) coefficients, of any $2n + 1$ members of the sequence $\{\exp(ikx)\}$, $-\infty < k < \infty$.

If n is a nonnegative integer and A_n is a set of $2n + 1$ distinct integers, we call a function $P(x)$ of the form

$$P(x) = \sum_{k \in A_n} a_k \exp(ikx), \tag{1}$$

where the a_k are complex constants, a *trigonometric n-nomial*. We investigate the degree of approximation to the space $L^2(-\pi, \pi)$ of complex-valued, square integrable functions on $(-\pi, \pi)$ by trigonometric n-nomials, and compare it to the degree of approximation by the set $\mathscr{T}_n$ of ordinary nth degree trigonometric polynomials.

For the purpose of comparison we concentrate our attention on the subset $\mathscr{S}$ of L^2 which consists of those $f \in L^2$ satisfying

$$\omega_f(h) \leqslant h \qquad \text{for all} \quad h \geqslant 0, \tag{2}$$

where $\omega_f(h)$ is the L^2 modulus of continuity of f,

$$\omega_f(h) = \sup_{|t| \leqslant h} \|f(x + t) - f(x)\|_{L^2}. \tag{3}$$

Throughout the paper, f will be an L^2 function with Fourier coefficients c_k, $-\infty < k < \infty$. We let $B_n(f)$ be the set of $2n + 1$ integers k where the maximum values of $|c_k|$ occur. In cases of equality, we take the k with smaller absolute value, and then (if necessary) we take $|k|$ before $-|k|$.

Given A_n, we let $P(A_n)$ denote the set of all functions of the form (1), and we let P_n denote the set of all trigonometric n-nomials (i.e., the set of all $P \in P(A_n)$ for all A_n). We define the following degrees of approximation:

(a) $E(f, A_n) = \inf_{P \in P(A_n)} \| f - P \|$,

(b) $\mathscr{E}(\mathscr{S}, A_n) = \sup_{f \in \mathscr{S}} E(f, A_n)$,

(c) $\mathscr{E}_n^*(\mathscr{S}) = \sup_{f \in \mathscr{S}} \inf_{T \in \mathscr{T}_n} \| f - T \|$,

(d) $D_n(f) = \inf_{P \in P_n} \| f - P \|$,

(e) $\mathscr{D}_n(\mathscr{S}) = \sup_{f \in \mathscr{S}} D_n(f)$.

Thus, we wish to compare $\mathscr{E}_n^*(\mathscr{S})$ and $\mathscr{D}_n(\mathscr{S})$.

Our principal tool for studying $\mathscr{S}$ will be the following well known lemma, which we prove for the sake of completeness.

LEMMA 1. *$f \in \mathscr{S}$ if and only if $\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1$.*

Proof. A straightforward calculation shows that

$$[\omega_f(h)]^2 = \sup_{|t| \leqslant h} 2 \sum_{-\infty}^{\infty} \mid c_k \mid^2 (1 - \cos kt).$$

Thus, we must show that

$$\sum_{-\infty}^{\infty} \mid c_k \mid^2 \frac{1 - \cos kt}{\frac{1}{2}h^2} \leqslant 1 \qquad \text{for all} \quad \mid t \mid \leqslant h$$

if and only if $\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1$.

First, suppose that $\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1$. Then, since $1 - \cos x \leqslant \frac{1}{2}x^2$ for all x,

$$\sum_{-\infty}^{\infty} \mid c_k \mid^2 \frac{1 - \cos kt}{\frac{1}{2}h^2} \leqslant \sum_{-\infty}^{\infty} \mid c_k \mid^2 \frac{\frac{1}{2}(kt)^2}{\frac{1}{2}h^2} = \sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \frac{t^2}{h^2}$$

$$\leqslant \sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1.$$

Suppose, on the other hand, that $\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 > 1$, so that $\sum_{A}^{B} k^2 \mid c_k \mid^2 = \rho > 1$ for two integers $A < B$.

If we let $\epsilon = \rho - 1$ and rewrite ρ as

$$\rho = \sum_A^B |c_k|^2 \frac{1 - (1 - \frac{1}{2}k^2h^2)}{\frac{1}{2}h^2}$$

$$= \sum_A^B |c_k|^2 \frac{1 - \cos kh}{\frac{1}{2}h^2} + \sum_A^B |c_k|^2 \frac{\cos kh - (1 - \frac{1}{2}k^2h^2)}{\frac{1}{2}h^2}$$

we see that, since

$$\lim_{h\to 0} \frac{\cos kh - (1 - \frac{1}{2}k^2h^2)}{\frac{1}{2}h^2} = 0,$$

we can take h small enough so that

$$0 < \sum_A^B |c_k|^2 \frac{\cos kh - (1 - \frac{1}{2}k^2h^2)}{\frac{1}{2}h^2} < \epsilon,$$

and then

$$\sum_{-\infty}^{\infty} |c_k|^2 \frac{1 - \cos kh}{\frac{1}{2}h^2} \geqslant \sum_A^B |c_k|^2 \frac{1 - \cos kh}{\frac{1}{2}h^2} > \rho - \epsilon = 1,$$

and Lemma 1 is proved.

For our first theorem we have (in our notation) the well known characterization of best approximation in L^2.

THEOREM 1. $[E(f, A_n)]^2 = \sum_{k\notin A_n} |c_k|^2$.

By applying Lemma 1, along with Theorem 1, we get the following theorem.

THEOREM 2. *If* $0 \in A_n$, *then* $\mathscr{E}(\mathscr{S}, A_n) = 1/\gamma$, *where* $\gamma = \gamma(A_n) = \min_{k\notin A_n} |k|$. *If* $0 \notin A_n$, *then* $\mathscr{E}(\mathscr{S}, A_n) = +\infty$.

Proof. The case when $0 \notin A_n$ is trivial, so assume that $0 \in A_n$. Let $f \in \mathscr{S}$. Then

$$1 \geqslant \sum_{-\infty}^{\infty} k^2 |c_k|^2 = \sum_{k\in A_n} k^2 |c_k|^2 + \sum_{k\notin A_n} k^2 |c_k|^2$$

$$\geqslant \sum_{k\notin A_n} k^2 |c_k|^2 \geqslant \gamma^2 \sum_{k\notin A_n} |c_k|^2$$

so that, for $f \in \mathscr{S}$,

$$[E(f, A_n)]^2 = \sum_{k \notin A_n} |c_k|^2 \leqslant 1/\gamma^2, \qquad \text{or} \quad \mathscr{E}(\mathscr{S}, A_n) \leqslant 1/\gamma,$$

and the upper bound can be achieved by taking $g(x) = \gamma^{-1} \exp(i\gamma x)$ if $\gamma = k$, or $g(x) = \gamma^{-1} \exp(-i\gamma x)$ if $\gamma = -k$.

By applying Theorem 2 to the set $A_n = \{0, \pm 1,..., \pm n\}$ we immediately obtain the following corollary.

COROLLARY. $\mathscr{E}_n{}^*(\mathscr{S}) = 1/(n + 1)$.

For our first result concerning the set P_n we have the following theorem.

THEOREM 3. $[D_n(f)]^2 = \sum_{k \notin B_n(f)} |c_k|^2$.

Proof. This follows immediately from Theorem 1 and the fact that $D_n(f) = \inf E(f, A_n)$, where the infimum is taken over all possible A_n.

We now have our main result, which gives the exact value of $\mathscr{D}_n(\mathscr{S})$.

THEOREM 4. $[\mathscr{D}_n(\mathscr{S})]^2 = 4/((3n + 1)(3n + 2))$.

Proof. In order to prove Theorem 4 we require two lemmas, the first of which follows.

LEMMA 2. *Let* $\mathscr{S}^* = \{g: g \in \mathscr{S}$ *and* $B_n(g) = \{0, \pm 1,..., \pm n\}\}$. *If* $f \in \mathscr{S}$, *then there is a* $g \in \mathscr{S}^*$ *such that* $D_n(g) = D_n(f)$.

Proof. Define the following sets:

$$P = \{k: |k| \leqslant n \text{ and } k \in B_n(f)\},$$

$$Q = \{k: |k| > n \text{ and } k \in B_n(f)\},$$

$$R = \{k: |k| \leqslant n \text{ and } k \notin B_n(f)\},$$

$$S = \{k: |k| > n \text{ and } k \notin B_n(f)\}.$$

Obviously Q and R contain the same number of integers, and we let $q \leftrightarrow r$ be a one-to-one correspondence between these two sets. We now define the function $g(x)$ by $g(x) \sim \sum_{-\infty}^{\infty} b_k \exp(ikx)$, where

$$b_k = \begin{cases} c_k & \text{if} \quad k \in P \quad \text{or} \quad k \in S, \\ c_q & \text{if} \quad k = r \in R, \\ c_r & \text{if} \quad k = q \in Q. \end{cases}$$

Clearly Theorem 3 implies that $D_n(g) = D_n(f)$, and

$$\sum_{-\infty}^{\infty} k^2 \mid b_k \mid^2 = \sum_{k\in P} k^2 \mid b_k \mid^2 + \sum_{k\in Q} k^2 \mid b_k \mid^2 + \sum_{k\in R} k^2 \mid b_k \mid^2 + \sum_{k\in S} k^2 \mid b_k \mid^2$$

$$= \sum_{k\in P} k^2 \mid c_k \mid^2 + \sum_{k\in S} k^2 \mid c_k \mid^2 + \sum_{q\in Q} q^2 \mid c_r \mid^2 + \sum_{r\in R} r^2 \mid c_q \mid^2$$

$$\leqslant \sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1,$$

where the first inequality follows from the fact that $\mid c_q \mid \geqslant \mid c_r \mid$ and $r^2 < q^2$, so that $r^2 \mid c_q \mid^2 + q^2 \mid c_r \mid^2 \leqslant r^2 \mid c_r \mid^2 + q^2 \mid c_q \mid^2$ for all $q = q(r) \in Q$ and $r = r(q) \in R$. Thus $g \in \mathscr{S}^*$, and the proof of Lemma 2 is complete.

Returning to Theorem 4, we see that $\mathscr{D}_n(\mathscr{S}) = \sup_{f\in\mathscr{S}^*} D_n(f)$. Therefore, by Theorem 3, we have $[\mathscr{D}_n(\mathscr{S})]^2 = \max \sum_{|k|>n} \mid c_k \mid^2$, where the maximum is taken over all sequences $\{c_k\}_{-\infty}^{\infty}$ satisfying

(i) $\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1$ and

(ii) $\mid j \mid \leqslant n < \mid m \mid$ implies $\mid c_m \mid \leqslant \mid c_j \mid$.

We let α be this maximum, and we let $\beta = \max \sum_{k>n} a_k^2$, where the maximum is taken over all sequences $\{a_k\}_1^{\infty}$ satisfying

(iii) $i \leqslant n < j$ implies $0 \leqslant a_j \leqslant a_i$ and

(iv) $\sum_1^{\infty} k^2 a_k^2 \leqslant 1$.

For our final lemma we have the following.

LEMMA 3. $\alpha = \beta$.

Proof. Given a sequence $\{c_k\}$ satisfying (i) and (ii), define $a_k \geqslant 0$ by $a_k^2 = \mid c_{-k} \mid^2 + \mid c_k \mid^2$, $k \geqslant 1$. Then

$$\sum_1^{\infty} k^2 a_k^2 = \sum_1^{\infty} k^2[\mid c_{-k} \mid^2 + \mid c_k \mid^2] = \sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 \leqslant 1,$$

and $1 \leqslant i < n < j$ implies $a_j^2 = \mid c_{-j} \mid^2 + \mid c_j \mid^2 \leqslant \mid c_{-i} \mid^2 + \mid c_i \mid^2 = a_i^2$. Furthermore, $\sum_{k>n} a_k^2 = \sum_{|k|>n} \mid c_k \mid^2$, so that $\beta \geqslant \alpha$.

On the other hand, suppose $\{a_k\}_1^{\infty}$ is a sequence satisfying (iii) and (iv). Define $\{c_k\}_{-\infty}^{\infty}$ by $c_{-k}^2 = c_k^2 = \frac{1}{2} a_k^2$ for $k \geqslant 1$, and $c_0 = \max_{1\leqslant k<\infty} a_k$. Then

$$\sum_{-\infty}^{\infty} k^2 \mid c_k \mid^2 = \sum_{-\infty}^{\infty} \tfrac{1}{2} k^2 a_{|k|}^2 = \sum_1^{\infty} k^2 a_k^2 \leqslant 1,$$

while $0 < |i| \leqslant n < |j|$ implies $c_j^2 = \frac{1}{2}a_{|j|}^2 \leqslant \frac{1}{2}a_{|i|}^2 = c_i^2$, and again $\sum_{k>n} a_k^2 = \sum_{|k|>n} c_k^2$. Thus, $\alpha \geqslant \beta$, so that $\alpha = \beta$, completing the proof of Lemma 3.

To conclude the proof of Theorem 4 it remains to evaluate $\beta = [\mathscr{D}_n(\mathscr{S})]^2$. Toward this end, we let $J = [\frac{1}{2}(n+1)]$, and we observe that

$$\sum_{k=1}^{n+J-1} k^2 > (J-1)(n+J)^2 \tag{4}$$

and

$$J(n+J+1)^2 > \sum_{k=1}^{n+J} k^2. \tag{5}$$

By (iv), we have

$$1 - \sum_{1}^{n} k^2 a_k^2 \geqslant \sum_{n+1}^{\infty} k^2 a_k^2 \geqslant \sum_{n+1}^{n+J} k^2 a_k^2 + (n+J+1)^2 \sum_{n+J+1}^{\infty} a_k^2. \tag{6}$$

Using (iii) and (4), we see that

$$\sum_{i=1}^{J} \frac{\sum_{k=1}^{n+J} k^2 - J(n+i)^2}{J} a_{n+i}^2 \leqslant \min_{1 \leqslant k \leqslant n} a_k^2 \sum_{i=1}^{J} \frac{\sum_{k=1}^{n+J} k^2 - J(n+i)^2}{J}$$

$$= \min_{1 \leqslant k \leqslant n} a_k^2 \sum_{k=1}^{n} k^2 \leqslant \sum_{k=1}^{n} k^2 a_k^2. \tag{7}$$

Combining (6) and (7) with (5) yields

$$1 \geqslant \sum_{i=1}^{J} \frac{\sum_{k=1}^{n+J} k^2 - J(n+i)^2}{J} a_{n+i}^2 + \sum_{k=n+1}^{n+J} k^2 a_k^2 + (n+J+1)^2 \sum_{k=n+J+1}^{\infty} a_k^2$$

$$= \sum_{i=1}^{J} \left[\frac{\sum_{k=1}^{n+J} k^2 - J(n+i)^2}{J} + (n+i)^2\right] a_{n+i}^2 + (n+J+1)^2 \sum_{k=n+J+1}^{\infty} a_k^2$$

$$= 1/J \sum_{k=1}^{n+J} k^2 \sum_{m=n+1}^{n+J} a_m^2 + (n+J+1)^2 \sum_{k=n+J+1}^{\infty} a_k^2$$

$$= 1/J \sum_{k=1}^{n+J} k^2 \sum_{m=n+1}^{\infty} a_m^2 + \left[(n+J+1)^2 - 1/J \sum_{k=1}^{n+J} k^2\right] \sum_{m=n+J+1}^{\infty} a_m^2$$

$$\geqslant 1/J \sum_{k=1}^{n+J} k^2 \sum_{m=n+1}^{\infty} a_m^2,$$

or

$$\sum_{k>n} a_k^2 \leqslant J\Big/\sum_{k=1}^{n+J} k^2. \tag{8}$$

Moreover, the upper bound in (8) can be attained by setting

$$a_i^2 = \begin{cases} 1\Big/\sum_{k=1}^{n+J} k^2, & \text{for} \quad 1 \leqslant i \leqslant n+J, \\ 0 & \text{for} \quad i > n+J. \end{cases}$$

It is clear that with this choice for $\{a_i\}$, (iii) and (iv) are satisfied, and

$$\sum_{k>n} a_k^2 = J\Big/\sum_{k=1}^{n+J} k^2.$$

Therefore,

$$\beta = [\mathscr{D}_n(\mathscr{S})]^2 = J\Big/\sum_{k=1}^{n+J} k^2,$$

and Theorem 4 now follows immediately from the formula

$$\sum_{k=1}^{N} k^2 = (N(N+1)(2N+1))/6.$$

Finally, combining the corollary to Theorem 2 with Theorem 4, we see that $\mathscr{D}_n(\mathscr{S})$ is asymptotic to $\frac{2}{3}\mathscr{E}_n{}^*(\mathscr{S})$, and our comparison of these two means of approximation in $\mathscr{S}$ is complete.